Quantum mechanics with time-dependent parameters

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Abstract. Composite bundles $Q \to \Sigma \to \mathbf{R}$, where $\Sigma \to \mathbf{R}$ is the parameter bundle, provide the adequate mathematical description of classical mechanics with time-dependent parameters. We show that the Berry's phase phenomenon is described in terms of connections on composite Hilbert space bundles.

I.

Smooth fiber bundles $Q \to \mathbf{R}$ over a time axis $\mathbf{R}$ provide the adequate formulation of classical time-dependent mechanics treated as a particular field theory [1, 2]. Let us consider a mechanical system depending on time-dependent parameters. These parameters can be seen as sections of some smooth fiber bundle $\Sigma \to \mathbf{R}$. Then the configuration space of a mechanical system with time-dependent parameters can be seen as the composite fiber bundle

$$Q \to \Sigma \to \mathbf{R}. \tag{1}$$

In classical mechanics $Q \to \Sigma$ is a smooth finite-dimensional fiber bundle. In quantum mechanics $Q \to \Sigma$ is a C^*-algebra fiber bundle or a Hilbert space fiber bundle [3].

The following two facts make the composite fiber bundle (1) useful for our purpose.

(i) Given a section h of a parameter bundle $\Sigma \to \mathbf{R}$, the pull-back bundle h^*Q over $\mathbf{R}$ describes a mechanical system under the fixed parameter functions $h(t)$.

(ii) Given a connection A_Σ on the fiber bundle $Q \to \Sigma$, the pull-back connection h^*A_Σ on the pull-back bundle $h^*Q \to \mathbf{R}$ depends in a certain way on the parameter functions $h(t)$, and characterizes the dynamics of a mechanical system with time-dependent parameters.

This work is devoted to quantum mechanics with classical parameters where connections on composite Hilbert space bundles play the role of Berry connections.

II.

Recall that by a smooth composite bundle is meant the composition of fiber bundles

$$Y \to \Sigma \to X, \tag{2}$$

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where $\pi_{Y\Sigma} : Y \to \Sigma$ and $\pi_{\Sigma X} : \Sigma \to X$ are smooth fiber bundles [3, 4]. It is provided with an atlas of fibered coordinates $(x^\lambda, \sigma^m, y^i)$, where (x^μ, σ^m) are fibered coordinates on the fiber bundle $\Sigma \to X$ and the transition functions $\sigma^m \to \sigma'^m(x^\lambda, \sigma^k)$ are independent of the fiber coordinates y^i.

Proposition 1: Given a composite fiber bundle (2), let h be a global section of the fiber bundle $\Sigma \to X$. Then the restriction

$$Y_h = h^* Y \tag{3}$$

of the fiber bundle $Y \to \Sigma$ to $h(X) \subset \Sigma$ is a subbundle $i_h : Y_h \hookrightarrow Y$ of the fiber bundle $Y \to X$.

Let us consider a connection

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + A^i_\lambda \partial_i) + d\sigma^m \otimes (\partial_m + A^i_m \partial_i) : Y \to J^1_\Sigma Y \tag{4}$$

on the fiber bundle $Y \to \Sigma$. Given a section h the fiber bundle $\Sigma \to X$, the connection A_Σ (4) induces the pull-back connection

$$A_h = i^*_h A_\Sigma = dx^\lambda \otimes [\partial_\lambda + ((A^i_m \circ h)\partial_\lambda h^m + (A \circ h)^i_\lambda)\partial_i] \tag{5}$$

on the pull-back bundle Y_h (3).

Note that, in quantum theory, one follows the notion of a connection phrased in algebraic terms as a connection on modules in comparison with the pure geometric one in classical theory. Here, we restrict our consideration to connecions on modules over the ring $C^\infty(X)$ of smooth real functions on a manifold X [3, 5].

Definition 2: A connection on a $C^\infty(X)$-module $\mathcal{S}$ assigns to each vector field τ on a manifold X an $\mathcal{S}$-valued first order differential operator $\nabla_\tau \in \mathrm{Diff}_1(\mathcal{S}, \mathcal{S})$ on $\mathcal{S}$ which obeys the Leibniz rule

$$\nabla_\tau(fs) = (\tau \rfloor df)s + f\nabla_\tau s, \quad f \in C^\infty(X), \quad s \in \mathcal{S}. \tag{6}$$

If $\mathcal{S}$ is a module of global sections of a smooth vector bundle $Y \to X$ over a manifold X, Definition 2 is equivalent to the familiar geometric definition of a connection on $Y \to X$.

III.

Let us consider a quantum mechanical systems depending on a finite number of real classical parameters given by sections of a smooth parameter bundle $\Sigma \to \mathbf{R}$. For the sake of simplicity, we fix a trivialization $\Sigma = \mathbf{R} \times Z$, coordinated by (t, σ^m). Although it may happen that the parameter bundle $\Sigma \to \mathbf{R}$ has no preferable trivialization, e.g., if one of parameters is a velocity of a reference frame.

Recall that, in the framework of algebraic quantum theory, a quantum system is characterized by a C^*-algebra A and a positive (hence, continuous) form ϕ on A which defines the representation π_ϕ of A in a Hilbert space E_ϕ with a cyclic vector ξ_ϕ such that

$$\phi(a) = \langle \pi_\phi(a)\xi_\phi | \xi_\phi \rangle, \qquad \forall a \in A.$$

One says that $\phi(a)$ is a mean value of the operator a in the state ξ_ϕ.

It should be emphasized that, in quantum mechanics, a time also plays the role of a classical parameter. Indeed, all relations between operators in quantum mechanics are simultaneous, while a computation of a mean value of an operator in a quantum state does not imply an integration over a time. It follows that, at each moment, we have a quantum system, but these quantum systems are different at different instants. Though they may be isomorphic to each other. This characteristic is extended to other classical parameters. Namely, we assign a C^*-algebra A_σ to each point $\sigma \in \Sigma$ of the parameter bundle Σ, and treat A_σ as a quantum system under fixed values (t, σ^m) of the parameters.

Remark 1: Let us emphasize that one should distinguish classical parameters from coordinates which a wave function can depend on. Let $\{A_q\}$ be a set of C^*-algebras parameterized by points of a locally compact topological space Q. Let all C^*-algebras A_q are isomorphic to each other and to some C^*-algebra A. We consider a locally trivial topological fiber bundle $P \to Q$ whose typical fiber is the C^*-algebra A, i.e., transition functions of this fiber bundle provide automorphisms of A. The set $P(Q)$ of continuous sections of this fiber bundle is a *-algebra with respect to fiberwise operations. Let us consider a subalgebra $A(Q) \subset P(Q)$ which consists of sections α of $P \to Q$ such that the real function $||\alpha(q)||$ vanishes at infinity of Q. For $\alpha \in A(Q)$, put

$$||\alpha|| = \sup_{q\in Q} ||\alpha(q)|| < \infty.$$

With this norm, $A(Q)$ is a C^*-algebra [6]. One can consider a quantum system characterized by this C^*-algebra. In this case, elements of the set Q are not classical parameters as follows. Given an element $q \in Q$, the assignment

$$A(Q) \ni \alpha \mapsto \alpha(q) \in A \tag{7}$$

is a C^*-algebra epimorphism. Let π be a representation of A. Then the assignment (7) yields a representation $\rho(\pi, q)$ of the C^*-algebra $A(Q)$. If π is an irreducible representation of the C^*-algebra A, then $\rho(\pi, q)$ is an irreducible representation of $A(Q)$. Moreover, the irreducible representations $\rho(\pi, q)$ and $\rho(\pi, q')$ of $A(Q)$ are not equivalent [6]. Therefore there is one-to-one correspondence (but not a homeomorphism) between the spectrum $\widehat{A(Q)}$ of the

C^*-algebra $A(Q)$ and the product $Q \times \widehat{A}$ of Q and the spectrum $\widehat{A}$ of the C^*-algebra A. It follows that one can find representations of the C^*-algebra $A(Q)$ among direct integrals of representations of A with respect to some measure on Q. Let μ be a positive measure of total mass 1 on the locally compact space Q , and let ϕ be a positive form on A. Then the function $q \mapsto \phi(\alpha(q))$, $\forall \alpha \in A(Q)$, is a μ-measurable, while the integral

$$\phi(\alpha) = \int \phi(\alpha(q))\mu(q)$$

provides a positive form on the C^*-algebra $A(Q)$. Roughly speaking, a computation of a mean value of an operator $\alpha \in A(Q)$ implies an integration with respect to some measure on Q in general. This is not the case of quantum systems depending on classical parameters $q \in Q$.

We simplify our consideration in order to single out the manifested Berry's phase phenomenon. Let us assume that all algebras C^*-algebras A_σ, $\sigma \in \Sigma$, are isomorphic to the von Neumann algebra $B(E)$ of bounded operators in some Hilbert space E, and consider a locally trivial Hilbert space bundle $\Pi \to \Sigma$ with the typical fiber E and smooth transition functions [7]. Smooth sections of $\Pi \to \Sigma$ constitute a module $\Pi(\Sigma)$ over the ring $C^\infty(\Sigma)$ of real functions on Σ. In accordance with Definition 2, a connection $\widetilde{\nabla}$ on $\Pi(\Sigma)$ assigns to each vector field τ on Σ a first order differential operator

$$\widetilde{\nabla}_\tau \in \mathrm{Diff}_1(\Pi(\Sigma), \Pi(\Sigma)) \tag{8}$$

which obeys the Leibniz rule

$$\widetilde{\nabla}_\tau(fs) = (\tau \rfloor df)s + f\widetilde{\nabla}_\tau s, \qquad s \in \Pi(\Sigma), \qquad f \in C^\infty(\Sigma).$$

Let τ be a vector field on Σ such that $dt\rfloor\tau = 1$. Given a trivialization chart of the Hilbert space bundle $\Pi \to \Sigma$, the operator $\widetilde{\nabla}_\tau$ (8) reads

$$\widetilde{\nabla}_\tau(s) = (\partial_t - i\mathcal{H}(t,\sigma^i))s + \tau^m(\partial_m - i\widehat{A}_m(t,\sigma^i))s, \tag{9}$$

where $\mathcal{H}(t,\sigma^i)$, $\widehat{A}_m(t,\sigma^i)$ for each $\sigma \in \Sigma$ are bounded self-adjoint operators in the Hilbert space E.

Let us consider the composite fiber bundle $\Pi \to \Sigma \to \mathbf{R}$. Similarly to the case of smooth composite fiber bundles (see Proposition 1), every section $h(t)$ of the parameter bundle $\Sigma \to \mathbf{R}$ defines the subbundle $\Pi_h = h^*\Pi \to \mathbf{R}$ of the composite fiber bundle $\Pi \to \mathbf{R}$ whose typical fiber is the Hilbert space E. Accordingly, the connection $\widetilde{\nabla}$ (9) on the $C^\infty(\Sigma)$-module $\Pi(\Sigma)$ defines the pull-back connection

$$\nabla_h(\psi) = [\partial_t - i(\widehat{A}_m(t,h^i(t))\partial_t h^m + \mathcal{H}(t,h^i(t))]\psi \tag{10}$$

on the $C^\infty(\mathbf{R})$-module $\Pi_h(\mathbf{R})$ of sections ψ of the Hilbert space bundle $\Pi_h \to \mathbf{R}$.

As in the case of smooth fiber bundles, we say that a section ψ of the fiber bundle $\Pi_h \to \mathbf{R}$ is an integral section of the connection (10) if

$$\nabla_h(\psi) = [\partial_t - i(\widehat{A}_m(t, h^i(t))\partial_t h^m + \mathcal{H}(t, h^i(t))]\psi = 0. \tag{11}$$

One can think of the equation (11) as being the Shrödinger equation for a quantum system depending on the parameter function $h(t)$. Its solutions take the form

$$G_t = T\exp\left[i\int_0^t (\widehat{A}_m \partial_{t'} h^m + \mathcal{H})dt'\right], \tag{12}$$

where G_t is the time-ordered exponent. The term $i\widehat{A}_m(t, h^i(t))\partial_t h^m$ in the Shrödinger equation (11) is responsible for the Berry's phase phenomenon, while $\mathcal{H}$ is treated as an ordinary Hamiltonian of a quantum system.

To show the Berry's phase phenomenon clearly, we simplify again the system under consideration. Given a trivialization of the fiber bundle $\Pi \to \mathbf{R}$ and the above mentioned trivialization $\Sigma = \mathbf{R} \times Z$ of the parameter bundle Σ, let us suppose that the components $\widehat{A}_m$ of the connection $\widetilde{\nabla}$ (9) are independent of t and that the operators $\mathcal{H}(\sigma)$ commute with the operators $\widehat{A}_m(\sigma)$ at all points of the curve $h(t) \subset \Sigma$. Then the operator G_t (12) takes the form

$$G_t = T\exp\left[i\int_{h([0,t])} \widehat{A}_m(\sigma^i)d\sigma^m\right] \cdot T\exp\left[i\int_0^t \mathcal{H}(t')dt'\right]. \tag{13}$$

One can think of the first factor in the right-hand side of the expression (13) as being the operator of a parallel transport along the curve $h([0,t]) \subset Z$ with respect to the pull-back connection

$$\nabla = i^*\widetilde{\nabla} = \partial_m - i\widehat{A}_m(t, \sigma^i) \tag{14}$$

on the fiber bundle $\Pi \to Z$, defined by the imbedding

$$i : Z \hookrightarrow \{0\} \times Z \subset \Sigma.$$

Note that, since operators $\widehat{A}_m$ are independent of time, one can utilize any imbedding of Z to $\{t\} \times Z$.

Moreover, the connection ∇ (14), called the Berry connection, can be seen as a connection on some principal fiber bundle $P \to Z$ for the group $U(E)$ of unitary operators in the Hilbert space E. Let the curve $h([0,t])$ be closed, while the holonomy group of the connection ∇ at the point $h(t) = h(0)$ is not trivial. Then the unitary operator

$$T\exp\left[i\int_{h([0,t])} \widehat{A}_m(\sigma^i)d\sigma^m\right] \tag{15}$$

is not the identity. For example, if

$$i\widehat{A}_m(\sigma^i) = iA_m(\sigma^i)\mathrm{Id}\,_E \tag{16}$$

is a $U(1)$-principal connection on Z, then the operator (15) is the well-known Berry phase factor

$$\exp\left[i\int\limits_{h([0,t])} A_m(\sigma^i)d\sigma^m\right].$$

If (16) is a curvature-free connection, Berry's phase is exactly the Aharonov–Bohm effect on the parameter space Z.

The following variant of the Berry's phase phenomenon leads us to a principal bundle for familiar finite-dimensional Lie groups. Let E be a separable Hilbert space which is the Hilbert sum of n-dimensional eigenspaces of the Hamiltonian $\mathcal{H}(\sigma)$, i.e.,

$$E = \bigoplus_{k=1}^{\infty} E_k, \qquad E_k = P_k(E),$$

where P_k are the projection operators, i.e.,

$$H(\sigma)\circ P_k = \lambda_k(\sigma)P_k$$

(in the spirit of the adiabatic hypothesis). Let the operators $\widehat{A}_m(z)$ be time-independent and preserve the eigenspaces E_k of the Hamiltonian $\mathcal{H}$, i.e.,

$$\widehat{A}_m(z) = \sum_k \widehat{A}^k_m(z)\circ P_k, \tag{17}$$

where $\widehat{A}^k_m(z)$, $z\in Z$, are self-adjoint operators in E_k. It follows that $\widehat{A}_m(\sigma)$ commute with $\mathcal{H}(\sigma)$ at all points of the parameter bundle $\Sigma\to\mathbf{R}$. Then, restricted to each subspace E_k, the parallel transport operator (15) is a unitary operator in E_k. In this case, the Berry connection (14) on the $U(E)$-principal bundle $P\to Z$ can be seen as a composite connection on the composite bundle

$$P\to P/U(n)\to Z,$$

which is defined by some principal connection on the $U(n)$-principal bundle $P\to P/U(n)$ and the trivial connection on the fiber bundle $P/U(n)\to Z$. The typical fiber of $P/U(n)\to Z$ is exactly the classifying space $B(U(n))$ for $U(n)$-principal bundles. Moreover, one can consider the parallel transport along a curve in the bundle $P/U(n)$. In this case, a state vector $\psi(t)$ acquires a geometric phase factor in addition to the dynamical phase factor. In particular, if $\Sigma=\mathbf{R}$ (i.e., classical parameters are absent and Berry's phase has only the geometric origin) we come to the case of a Berry connection on the $U(n)$-principal bundle on the classifying space $B(U(n))$ [8]. If $n=1$, this is the variant of Berry's geometric phase of Ref. [9].

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